Homework 12 (Due 4/30/2014)

Math 622

May 13, 2014

1. For this problem, the result of Exercise 5.4 in Shreve is useful.

Suppose the risk-neutral model for a risky asset $S(t)$, $t \leq 3$, and a zero-coupon bond $B(t,3)$ maturing at $T = 3$ is

$$\begin{aligned} dS(t) &= R(t)S(t)\,dt + \sigma S(t)\,d\widetilde{W}(t), \quad t \leq 3; & (1) \\ dB(t,3) &= R(t)B(t,3)\,dt + (3-t)B(t,3)\,d\widetilde{W}(t), \quad t \leq 3. & (2) \end{aligned}$$

Let $\mathrm{For}_S(t) = \dfrac{S(t)}{B(t,3)}$ be the $\{T = 3\}$-forward price of $S(t)$.

Let $\widetilde{\mathbf{P}}^{(3)}$ denote the risk-neutral measure for the numéraire $B(t,3)$.

Write down the model under the risk-neutral measure $\widetilde{\mathbf{P}}^{(3)}$ for the numéraire $B(t,3)$ and use it to find an explicit formula for the price of a European call, $V(0) = \tilde{E}\Big[D(3)(S(3)-K)^+|S(0) = s_0]$, in terms of s_0 and $B(0,s)$.

2. Shreve, Exercise 10.9

3. In each case, determine if the model for the forward rate satisfies the HJM no-arbitrage conditions, and, if it does, determine the market price of risk.

a) $df(t,T) = \Big[\dfrac{T^3t^2}{2} + 5Tt - \dfrac{Tt^4}{2}\Big]\,dt + Tt\,dW(t)$.

b) $df(t,T) = [T - t - 2Tt^2]\,dt + dW_1(t) + 2T dW_2(t)$.

4. Shreve, Exercise 10.11. Note that the swap consists of series of payments, coming at times $T_1, T_2, \ldots, T_{n+1}$, where $T_j = j\delta$. You are to find the value of the swap at time 0, which is the sum of the values of each individual payment.

5. Read the derivation of Black's caplet formula, Theorem 10.4.2. This problem is a variation on the same theme. Let $L(t,T)$ denote forward LIBOR, as defined in Shreve, section 10.4. In the last class, we showed that if the risk-neutral HJM model for the zero-coupon bond price is

$$dB(t,T) = R(t)B(t,T)\,dt - \sigma^*(t,T)B(t,T)\,d\widetilde{W}(t), \qquad 0 < t \le T \le \bar{T}, \tag{3}$$

then forward LIBOR, $L(t,T)$, will solve (10.4.9) for $0 \le t \le T \le \bar{T} - \delta$, where $\gamma(t,T)$ is given by equation (10.4.15). This result is derived directly in Shreve in section 10.4.5; we derived it more directly in class by applying Theorem 9.2.

Consider the one factor Hull-White model with constant coefficients for interest rate $R(t)$ (also called the Vasicek model). This defines a particular risk-neutral HJM model of the form (3) for zero-coupon bond prices. The formula for σ is presented in section 10.3.5. Derive the explicit form of γ and hence of (10.4.9). Solve this equation explicitly for $L(t,T)$ and show that there is a variation of Black's caplet formula that is valid for pricing caplets in this situation, even though $\gamma(t,T)$ is now random. (You should find a simple stochastic d.e. for $d[c + L(t,T)]$ for some constant c. Use the result of exercise 5.4 in the text.)